ON SIMPLE LABELLED GRAPH C^*-ALGEBRAS

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Abstract. We consider the simplicity of the C^*-algebra associated to a labelled space $(E,\mathcal{L},\overline{\mathcal{E}})$, where $(E,\mathcal{L})$ is a labelled graph and $\overline{\mathcal{E}}$ is the smallest accommodating set containing all generalized vertices. We prove that if $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ is simple, then $(E,\mathcal{L},\overline{\mathcal{E}})$ is strongly cofinal, and if, in addition, $\{v\}\in\overline{\mathcal{E}}$ for every vertex v, then $(E,\mathcal{L},\overline{\mathcal{E}})$ is disagreeable. It is observed that $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ is simple whenever $(E,\mathcal{L},\overline{\mathcal{E}})$ is strongly cofinal and disagreeable, which is recently known for the C^*-algebra $C^*(E,\mathcal{L},\mathcal{E}^{0,-})$ associated to a labelled space $(E,\mathcal{L},\mathcal{E}^{0,-})$ of the smallest accommodating set $\mathcal{E}^{0,-}$.

1. Introduction

Given a directed graph $E=(E^0,E^1)$ with the vertex set E^0 and the edge set E^1, the C^*-algebra $C^*(E)$ generated by the family of universal Cuntz-Krieger E-family is associated (see [6, 7, 8] among others). A Cuntz-Kriger algebra can now be viewed as a graph C^*-algebra of a finite graph. It is well known that a graph C^*-algebra of a row finite graph E is simple if and only if the graph E is cofinal and every loop has an exit ([6]).

Let E be a directed graph and $\mathcal{A}$ be a countable alphabet. If $\mathcal{L}:E^1\to\mathcal{A}$ is a labelling map, we call $(E,\mathcal{L})$ a labelled graph. If $\mathcal{B}\subset 2^{E^0}$ is an accommodating set for $(E,\mathcal{L})$, there exists a labelled graph C^*-algebra $C^*(E,\mathcal{L},\mathcal{B})$ associated to the labelled space $(E,\mathcal{L},\mathcal{B})$ ([3]). Graph C^*-algebras and more generally ultragraph C^*-algebras ([9, 10]) are labelled graph C^*-algebras ([3]).

The simplicity of a labelled graph C^*-algebra $C^*(E,\mathcal{L},\mathcal{B})$ is considered in [4] when $\mathcal{B}$ is the smallest accommodating set $\mathcal{E}^{0,-}$. Using the generalized vertices that play the role of the vertices in a directed graph one has the useful expression of the elements of a dense set in $C^*(E,\mathcal{L},\mathcal{E}^{0,-})$. But a generalized vertex does not necessarily belong to the accommodating set $\mathcal{E}^{0,-}$. In this paper we consider the simplicity of the labelled graph C^*-algebra $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ where $\overline{\mathcal{E}}$ is the smallest accommodating set containing the generalized vertices.

We first review in the next section the definition of a graph C^*-algebra (from [6, 7]) and a labelled graph C^*-algebra (from [3, 4]) to set up the notations. Then we show in section 3 that if a labelled graph C^*-algebra

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$C^*(E,\mathcal{L},\overline{\mathcal{E}})$ is simple, the labelled space $(E,\mathcal{L},\overline{\mathcal{E}})$ is strongly cofinal (Theorem 3.8). If, in addition, $\{v\}\in\overline{\mathcal{E}}$ for each $v\in E^0$, it is shown that $(E,\mathcal{L},\overline{\mathcal{E}})$ is disagreeable (Theorem 3.14). The fact that if a labelled space $(E,\mathcal{L},\overline{\mathcal{E}})$ is strongly cofinal and disagreeable then $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ is simple can be obtained by a slight modification of the proof of [4, Theorem 6.4] (Theorem 3.16).

2. Labelled spaces and their C^*-algebras $C^*(E,\mathcal{L},\mathcal{B})$

A *directed graph* $E=(E^0,E^1,r,s)$ consists of the vertex set E^0, the edge set E^1, and the range, source maps $r_E, s_E : E^1\to E^0$. We shall simply refer to directed graphs as graphs and often write r, s for r_E, s_E. By E^n we denote the set of all finite paths $\lambda=\lambda_1\cdots\lambda_n$ of *length* n ($|\lambda|=n$), ($\lambda_i\in E^1,\ r(\lambda_i)=s(\lambda_{i+1}), 1\le i\le n-1$). We also set $E^{\le n}:=\cup_{i=1}^n E^i$ and $E^{\ge n}:=\cup_{i=n}^\infty E^i$. The maps r and s naturally extend to $E^{\ge 1}$. Infinite paths $\lambda_1\lambda_2\lambda_3\cdots$ can also be considered ($r(\lambda_i)=s(\lambda_{i+1})$, $\lambda_i\in E^1$) and we denote the set of all infinite paths by E^∞ extending s to E^∞ by $s(\lambda_1\lambda_2\lambda_3\cdots):=s(\lambda_1)$.

A *labelled graph* $(E,\mathcal{L})$ over a countable alphabet set $\mathcal{A}$ consists of a directed graph E and a *labelling map* $\mathcal{L}:E^1\to\mathcal{A}$. We assume that $\mathcal{L}$ is onto. For a finite path $\lambda=\lambda_1\cdots\lambda_n\in E^n$, put $\mathcal{L}(\lambda)=\mathcal{L}(\lambda_1)\cdots\mathcal{L}(\lambda_n)$. Similarly, we put $\mathcal{L}(E^\infty)=\{\mathcal{L}(\lambda_1)\mathcal{L}(\lambda_2)\cdots \mid \lambda_1\lambda_2\cdots\in E^\infty\}$. For a path $\alpha=\alpha_1\cdots\alpha_n\in E^{\ge 1}\cup\mathcal{L}(E^{\ge 1})$ and $1\le k\le l\le n<\infty$, we set $\alpha_{[k,l]}:=\alpha_k\cdots\alpha_l$ (and $\alpha_{[k,\infty)}:=\alpha_k\alpha_{k+1}\cdots$ for $\alpha\in E^\infty\cup\mathcal{L}(E^\infty)$). The *range* and *source* of a labelled path $\alpha\in\mathcal{L}(E^{\ge 1})$ are subsets of E^0 defined by

$$r_{\mathcal{L}_E}(\alpha)=\{r(\lambda)\mid \lambda\in E^{\ge 1},\ \mathcal{L}(\lambda)=\alpha\},\ s_{\mathcal{L}_E}(\alpha)=\{s(\lambda)\mid \lambda\in E^{\ge 1},\ \mathcal{L}(\lambda)=\alpha\}.$$

(For both a labelled graph and its underlying graph, we usually use the same notation r and s to denote range maps r_E, $r_{\mathcal{L}_E}$ and source maps s_E, $s_{\mathcal{L}_E}$.) The *relative range of* $\alpha\in\mathcal{L}(E^{\ge 1})$ *with respect to* $A\subset 2^{E^0}$ is defined to be

$$r(A,\alpha)=\{r(\lambda)\mid \lambda\in E^{\ge 1},\ \mathcal{L}(\lambda)=\alpha,\ s(\lambda)\in A\}.$$

If $\mathcal{B}\subset 2^{E^0}$ is a collection of subsets of E^0 such that $r(A,\alpha)\in\mathcal{B}$ whenever $A\in\mathcal{B}$ and $\alpha\in\mathcal{L}(E^{\ge 1})$, $\mathcal{B}$ is said to be *closed under relative ranges* for $(E,\mathcal{L})$. We call $\mathcal{B}$ an *accommodating set* for $(E,\mathcal{L})$ if it is closed under relative ranges, finite intersections and unions and contains $r(\alpha)$ for all $\alpha\in\mathcal{L}(E^{\ge 1})$. If $\mathcal{B}$ is accommodating for $(E,\mathcal{L})$, the triple $(E,\mathcal{L},\mathcal{B})$ is called a *labelled space*. A labelled space $(E,\mathcal{L},\mathcal{B})$ is *weakly left-resolving* if

$$r(A,\alpha)\cap r(B,\alpha)=r(A\cap B,\alpha)$$

for every $A,B\in\mathcal{B}$ and every $\alpha\in\mathcal{L}(E^{\ge 1})$. If $(E,\mathcal{L},\mathcal{B})$ is weakly left-resolving and A, B, $A\setminus B\in\mathcal{B}$, then it follows that for $\alpha\in\mathcal{L}(E^{\ge 1})$

$$r(A\setminus B,\alpha)=r(A,\alpha)\setminus r(B,\alpha). \tag{1}$$

But (1) may not hold if $A\setminus B\notin\mathcal{B}$ as we see in Example 2.4. For $A,B\in 2^{E^0}$ and $n\ge 1$, let

$$AE^n=\{\lambda\in E^n\mid s(\lambda)\in A\},\quad E^nB=\{\lambda\in E^n\mid r(\lambda)\in B\},$$

and $AE^nB = AE^n \cap E^nB$. We write E^nv for $E^n\{v\}$ and vE^n for $\{v\}E^n$, and will use notations like $AE^{\geq k}$ and vE^∞ which should have obvious meaning. A labelled graph $(E,\mathcal{L})$ is *left-resolving* if for all $v \in E^0$ the map $\mathcal{L} : E^1v \to \mathcal{L}(E^1v)$ is bijective (hence $\mathcal{L} : E^{\leq k}v \to \mathcal{L}(E^{\leq k}v)$ is bijective for all $k \geq 1$). A labelled space $(E,\mathcal{L},\mathcal{B})$ is said to be *set-finite* (*receiver set-finite*, respectively) if for every $A \in \mathcal{B}$ the set $\mathcal{L}(AE^1)$ ($\mathcal{L}(E^1A)$, respectively) finite.

Definition 2.1. ([3, Definition 4.1]) Let $(E,\mathcal{L},\mathcal{B})$ be a weakly left-resolving labelled space. A *representation* of $(E,\mathcal{L},\mathcal{B})$ consists of projections $\{p_A : A \in \mathcal{B}\}$ and partial isometries $\{s_a : a \in \mathcal{A}\}$ such that for $A, B \in \mathcal{B}$ and $a, b \in \mathcal{A}$,

(i) $p_{A\cap B} = p_Ap_B$ and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$, where $p_\emptyset = 0$,
(ii) $p_As_a = s_ap_{r(A,a)}$,
(iii) $s_a^*s_a = p_{r(a)}$ and $s_a^*s_b = 0$ unless $a = b$,
(iv) for $A \in \mathcal{B}$, if $\mathcal{L}(AE^1)$ is finite and non-empty, then

$$p_A = \sum_{a\in\mathcal{L}(AE^1)} s_ap_{r(A,a)}s_a^*.$$

It is known [3, Theorem 4.5] that if $(E,\mathcal{L},\mathcal{B})$ is a weakly left-resolving labelled space, there exists a C^*-algebra $C^*(E,\mathcal{L},\mathcal{B})$ generated by a universal representation $\{s_a, p_A\}$ of $(E,\mathcal{L},\mathcal{B})$. We call $C^*(E,\mathcal{L},\mathcal{B})$ the *labelled graph C^*-algebra* of a labelled space $(E,\mathcal{L},\mathcal{B})$.

Assumption 2.2. Throughout this paper, we assume the following.

(a) E has no sinks, that is $|s^{-1}(v)| > 0$ for all $v \in E^0$, and $(E,\mathcal{L},\mathcal{B})$ is set-finite and receiver set-finite.
(b) If $e, f \in E^1$ are edges with $s(e) = s(f)$, $r(e) = r(f)$, and $\mathcal{L}(e) = \mathcal{L}(f)$, then $e = f$.

Remark 2.3. Let $C^*(E,\mathcal{L},\mathcal{B})$ be the labelled graph C^*-algebra of $(E,\mathcal{L},\mathcal{B})$ with generators $\{s_a, p_A\}$. Then $s_a \neq 0$ and $p_A \neq 0$ for $a \in \mathcal{A}$ and $A \in \mathcal{B}$, $A \neq \emptyset$. Note also that $s_\alpha p_A s_\beta^* \neq 0$ if and only if $A \cap r(\alpha) \cap r(\beta) \neq \emptyset$. Since we assume that $(E,\mathcal{L},\mathcal{B})$ is set-finite and E has no sinks, by [3, Lemma 4.4] and Definition 2.1(iv) it follows that

$$p_A = \sum_{\sigma\in\mathcal{L}(AE^n)} s_\sigma p_{r(A,\sigma)}s_\sigma^* \ \text{ for } A \in \mathcal{B} \text{ and } n \geq 1$$

and

$$C^*(E,\mathcal{L},\mathcal{B}) = \overline{span}\{s_\alpha p_A s_\beta^* \mid \alpha,\, \beta \in \mathcal{L}(E^{\geq 1}),\ A \in \mathcal{B}\}.$$

For $v, w \in E^0$, we write $v \sim_l w$ if $\mathcal{L}(E^{\leq l}v) = \mathcal{L}(E^{\leq l}w)$ as in [4]. Then $\sim_l$ is an equivalence relation on E^0. The equivalence class $[v]_l$ of v is called a *generalized vertex*. Let $\Omega_l := E^0/\sim_l$. For $k < l$ and $v \in E^0$, $[v]_l \subset [v]_k$

is obvious and $[v]_l = \cup_{i=1}^m [v_i]_{l+1}$ for some vertices $v_1, \ldots, v_m \in [v]_l$ ([4, Proposition 2.4]).

Let $\mathcal{E}^{0,-}$ be the smallest accommodating set for $(E, \mathcal{L})$. Then

$$\mathcal{E}^{0,-} = \{\cup_{k=1}^m \cap_{i=1}^n r(\beta_{i,k}) \mid \beta_{i,k} \in \mathcal{L}(E^{\geq 1})\}$$

(see [4, Remark 2.1]). Also every accommodating set $\mathcal{B}$ for $(E, \mathcal{L})$ contains $\mathcal{E}^{0,-}$. By [4, Proposition 2.4], every generalized vertex $[v]_l$ is the difference of two sets $X_l(v)$ and $r(Y_l(v))$ in $\mathcal{E}^{0,-}$, more precisely,

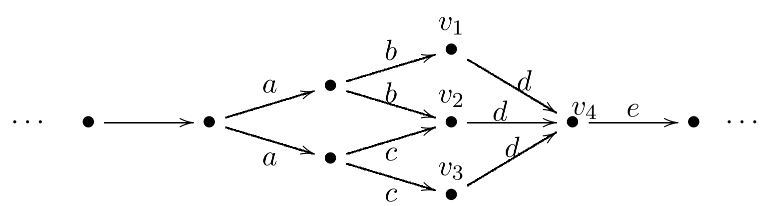
$$[v]_l = X_l(v) \setminus r(Y_l(v)),$$

where $X_l(v) := \cap_{\alpha \in \mathcal{L}(E^{\leq l} v)} r(\alpha)$ and $Y_l(v) := \cup_{w \in X_l(v)} \mathcal{L}(E^{\leq l} w) \setminus \mathcal{L}(E^{\leq l} v)$. The following example shows that

$$r([v]_l, a) \neq r(X_l(v), a) \setminus r(r(Y_l(v)), a),$$

in general.

Example 2.4. Consider the following weakly left-resolving labelled space $(E, \mathcal{L}, \mathcal{E}^{0,-})$:



Since $\{v_1, v_2\} = r(ab)$, $\{v_2, v_3\} = r(ac)$, and $\{v_2\} = \{v_1, v_2\} \cap \{v_2, v_3\}$, we have $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_2\} \in \mathcal{E}^{0,-}$. From $X_2(v_1) = \{v_1, v_2\}$, $Y_2(v_1) = \{c, ac\}$ and $r(Y_2(v_1)) = \{v_2, v_3\}$, we have

$$r(X_2(v_1), d) \setminus r(r(Y_2(v_1)), d) = \{v_4\} \setminus \{v_4\} = \emptyset.$$

But $X_2(v_1) \setminus r(Y_2(v_1)) = \{v_1\} = [v_1]_2 \notin \mathcal{E}^{0,-}$ and $r([v_1]_2, d) = r(\{v_1\}, d) = \{v_4\} \neq \emptyset$.

3. Simplicity of a labelled graph C^*-algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$

Recall ([4]) that a labelled space $(E, \mathcal{L}, \mathcal{B})$ is *l-cofinal* if for all $x \in \mathcal{L}(E^\infty)$, $[v]_l \in \Omega_l$, and $w \in s(x)$, there are $R(w) \geq l$, $N \geq 1$, and finitely many labelled paths $\lambda_1, \ldots, \lambda_m$ such that for all $d \geq R(w)$ we have

$$r([w]_d, x_{[1,N]}) \subseteq \cup_{i=1}^m r([v]_l, \lambda_i)$$

Since $[w]_{d'} \subset [w]_d$ whenever $d \leq d'$, we may restate the definition of an l-cofinal labelled space as follows.

Definition 3.1. Let $(E, \mathcal{L}, \mathcal{B})$ be a labelled space.

(a) ([4]) $(E, \mathcal{L}, \mathcal{B})$ is l-cofinal if for all $x \in \mathcal{L}(E^\infty)$, $[v]_l \in \Omega_l$, and $w \in s(x)$, there are $d \geq l$, $N \geq 1$, and a finite number of labelled paths $\lambda_1, \ldots, \lambda_m$ such that

$$r([w]_d, x_{[1,N]}) \subseteq \cup_{i=1}^m r([v]_l, \lambda_i).$$

$(E, \mathcal{L}, \mathcal{B})$ is *cofinal* if there is an $L > 0$ such that $(E, \mathcal{L}, \mathcal{B})$ is l-cofinal for all $l \geq L$.

(b) $(E, \mathcal{L}, \mathcal{B})$ is *strongly cofinal* if for all $x \in \mathcal{L}(E^\infty)$, $[v]_l \in \Omega_l$, and $w \in s(x)$, there are $N \geq 1$ and a finite number of labelled paths $\lambda_1, \ldots, \lambda_m$ such that

$$r([w]_1, x_{[1,N]}) \subset \cup_{i=1}^m r([v]_l, \lambda_i).$$

Every strongly cofinal labelled space is cofinal. Note that $(E, \mathcal{L}, \mathcal{B})$ is not cofinal if and only if there is a sequence $l_1 < l_2 < \cdots$ of positive integers such that $(E, \mathcal{L}, \mathcal{B})$ is not l_i-cofinal for all $i \geq 1$. But, in fact, we have the following.

Proposition 3.2. *A labelled space $(E, \mathcal{L}, \mathcal{B})$ is cofinal if and only if it is l-cofinal for all $l \geq 1$.*

Proof. If the labelled space is l-cofinal for all $l \geq 1$, obviously it is cofinal. For the converse, it suffices to show that if $(E, \mathcal{L}, \mathcal{B})$ is not l-cofinal, it is not l'-cofinal for all $l' > l$. To see this, first note that $(E, \mathcal{L}, \mathcal{B})$ is not l-cofinal if and only if the set $\triangle_l$ of triples $(x, [v]_l, w)$ of $x \in \mathcal{L}(E^\infty)$, $[v]_l \in \Omega_l$, and $w \in s(x)$ such that

$$r([w]_d, x_{[1,N]}) \nsubseteq \cup_{i=1}^m r([v]_l, \lambda_i)$$

for all $d \geq l$, $N \geq 1$, and a finite number of labelled paths $\lambda_1, \ldots, \lambda_m$ is nonempty. Suppose $(E, \mathcal{L}, \mathcal{B})$ is not l-cofinal and $(x, [v]_l, w) \in \triangle_l$. If $(E, \mathcal{L}, \mathcal{B})$ is l'-cofinal for $l' > l$, then there are $d \geq l'$, $N \geq 1$, and a finite number of labelled paths $\lambda_1, \ldots, \lambda_m$ such that

$$r([w]_d, x_{[1,N]}) \subseteq \cup_i r([v]_{l'}, \lambda_i).$$

But this contradicts to $(x, [v]_l, w) \in \triangle_l$ since $\cup_i r([v]_{l'}, \lambda_i) \subseteq \cup_i r([v]_l, \lambda_i)$. ☐

Notation 3.3. Let $\overline{\mathcal{E}}$ denote the set of all finite unions of generalized vertices. Then by [4, Proposition 2.4],

$$\mathcal{E}^{0,-} \subset \overline{\mathcal{E}} = \{\cup_{i=1}^m [v_i]_l \mid v_i \in E^0,\ m,\, l \geq 1\}.$$

For the proof of the following proposition it is helpful to note that if $[v]_l \cap [w]_k \neq \emptyset$, then either $[v]_l \subset [w]_k$ or $[w]_k \subset [v]_l$.

Proposition 3.4. *Let $(E, \mathcal{L})$ be a labelled graph such that*

$$r(A, \alpha) \cap r(B, \alpha) = r(A \cap B, \alpha) \tag{2}$$

for all $A, B \in \overline{\mathcal{E}}$ and $\alpha \in \mathcal{L}(E^{\geq 1})$. Then $\overline{\mathcal{E}}$ is a weakly left-resolving accommodating set such that $A \setminus B \in \overline{\mathcal{E}}$ for $A, B \in \overline{\mathcal{E}}$. If $(E, \mathcal{L}, \mathcal{E}^{0,-})$ is weakly left-resolving and $[v]_l \in \mathcal{E}^{0,-}$ for all $v \in E^0$ and $l \geq 1$, then $\overline{\mathcal{E}} = \mathcal{E}^{0,-}$.

Proof. Clearly $\overline{\mathcal{E}}$ is closed under finite intersections and unions. We show that $\overline{\mathcal{E}}$ is closed under relative ranges. For this, note first that $A \setminus B \in \overline{\mathcal{E}},\ \ A, B \in \mathcal{E}^{0,-}$ ([4, Proposition 2.4]). Since the two sets $[v]_l$ and $r(Y_l(v))(\in \mathcal{E}^{0,-})$ are disjoint, if $s(\alpha) \cap [v]_l \neq \emptyset$ and $s(\alpha) \cap r(Y_l(v)) \neq \emptyset$, then by (2),

$$r([v]_l, \alpha) \cap r(r(Y_l(v)), \alpha) = r([v]_l \cap r(Y_l(v)), \alpha) = \emptyset.$$

Hence $r([v]_l, \alpha) = r(X_l(v), \alpha) \setminus r(r(Y_l(v)), \alpha) \in \overline{\mathcal{E}}$ since both $r(X_l(v), \alpha)$ and $r(r(Y_l(v)), \alpha)$ belong to $\mathcal{E}^{0,-}$. For an arbitrary $C = \cup_{i=1}^n [v_i]_l \in \overline{\mathcal{E}}$,

$$r(C, \alpha) = r(\cup_{i=1}^n [v_i]_l, \alpha) = \cup_{i=1}^n r([v_i]_l, \alpha) \in \overline{\mathcal{E}}.$$

□

The labelled space $(E, \mathcal{L}, \mathcal{E}^{0,-})$ in Example 2.4 is weakly left-resolving, but $(E, \mathcal{L}, \overline{\mathcal{E}})$ is not weakly left-resolving since for $\{v_1\} = [v_1]_2, \in \overline{\mathcal{E}}$, $\{v_3\} = [v_3]_2 \in \overline{\mathcal{E}}$ we have

$$r(\{v_1\} \cap \{v_3\}, d) = \emptyset \neq \{v_4\} = r(\{v_1\}, d) \cap r(\{v_3\}, d).$$

We will consider only weakly left-resolving labelled spaces $(E, \mathcal{L}, \overline{\mathcal{E}})$ for the rest of this paper, so Example 2.4 is excluded from our discussion.

Recall that a graph E is *locally finite* if every vertex receives and emits only finite number of edges.

Proposition 3.5. *Let E be a locally finite graph and $(E, \mathcal{L}, \overline{\mathcal{E}})$ be a weakly left-resolving labelled space such that $|r(a)| = \infty$ for an $a \in \mathcal{A}$. Suppose that for each $v \in E^0$, $[v]_l$ is finite for some $l \geq 1$. Then the projection $p_{[v]_l}$ of a finite set $[v]_l$ generates a proper ideal of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$.*

Proof. Let I be the ideal of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ generated by the projection $p_{[v]_l}$. Suppose $I = C^*(E, \mathcal{L}, \overline{\mathcal{E}})$. Then there exists an $X \in I$ such that $\|s_a^* s_a - X\| < 1$. Let $X = \sum_{i=1}^m c_i (s_{\alpha_i} p_{A_i} s_{\beta_i}^*) p_{[v]_l} (s_{\gamma_i} p_{B_i} s_{\delta_i}^*)$, $c_i \in \mathbb{C}$. Then

$$V = \cup_{i=1}^m r([v]_l, \beta_i) \ \cup\ \cup_{i=1}^m r([v]_l, \gamma_i)$$

is a finite set since E is locally finite, and $V \in \overline{\mathcal{E}}$. Hence the set

$$W := \{s(\sigma_i) \mid \mathcal{L}(\sigma_i) = \alpha_i,\ r(\sigma_i) \cap V \neq \emptyset,\ \sigma_i \in E^{\geq 1},\ i = 1, 2, \ldots, m\}$$

is also finite. Since $r(a) \setminus V\ (\in \overline{\mathcal{E}})$ is an infinite set, one can choose $w \in r(a) \setminus V$ and $k \geq 1$ such that $[w]_k \cap W = \emptyset$. Then $r([w]_k, \alpha_i) \cap r([v]_l, \beta_j) = \emptyset$ for all $i, j = 1, \ldots, m$ and $p_{[w]_k} \leq p_{r(a)} = s_a^* s_a$. Thus

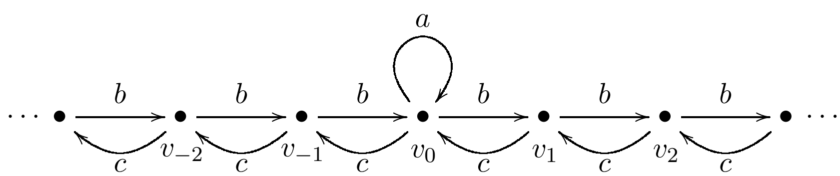
$$p_{[w]_k} (s_{\alpha_i} p_{A_i} s_{\beta_i}^*) p_{[v]_l} (s_{\gamma_i} p_{B_i} s_{\delta_i}^*) = s_{\alpha_i} p_{r([w]_k, \alpha_i) \cap A_i \cap r([v]_l, \beta_i)} s_{\beta_i}^* (s_{\gamma_i} p_{B_i} s_{\delta_i}^*) = 0$$

for all i, and then

$$1 > \ \|s_a^* s_a - X\| \geq \|p_{[w]_k} (s_a^* s_a - X) p_{[w]_k}\| = \|p_{[w]_k}\| = 1,$$

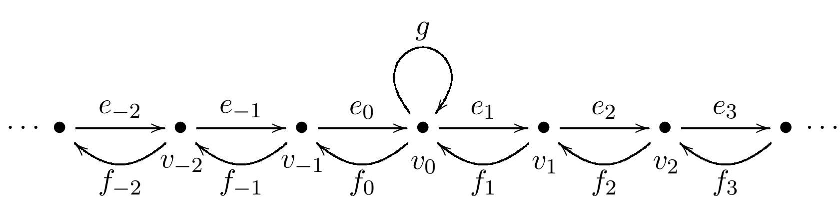
a contradiction. □

Example 3.6. Consider the following labelled graph $(E, \mathcal{L})$ of [4, 7.2]:



For each vertex v_k, we have $\{v_k\} = r(ab^k)$ for $k \geq 0$ and $\{v_k\} = r(ac^k)$ for $k < 0$. Hence $[v_k]_{k+1} = \{v_k\} \in \mathcal{E}^{0,-}$ for all k. Despite the fact that $\mathcal{E}^{0,-} = \{A \subset E^0 : A = E^0$ or A is a finite set$\}$ and $\overline{\mathcal{E}} = \mathcal{E}^{0,-} \cup \{B \subset E^0 : E^0 \setminus B$ is a finite set$\}$, one can show that $C^*(E, \mathcal{L}, \mathcal{E}^{0,-}) \cong C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ applying universal property of $C^*(E, \mathcal{L}, \mathcal{E}^{0,-})$ (since $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ contains a family of generators that is a representation of $C^*(E, \mathcal{L}, \mathcal{E}^{0,-})$) and the gauge invariant uniqueness theorem ([3, Theorem 5.3]). Since $|r(b)| = \infty$, by Proposition 3.5, $C^*(E, \mathcal{L}, \mathcal{E}^{0,-}) \cong C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is not simple (see Remark 3.15). But $(E, \mathcal{L}, \mathcal{E}^{0,-})$ is cofinal and disagreeable (we will discuss disagreeable labelled spaces later). Note that $(E, \mathcal{L}, \mathcal{E}^{0,-})$ is not strongly cofinal. In fact, for $[v_1]_1 = \{v_k \mid k \neq 0\}$, $[v_0]_1 = \{v_0\}$, and $x = bbb \cdots \in \mathcal{L}(E^\infty)$, the set $r([v_1]_1, x_{[1,N]})$ is infinite for any $N \geq 1$ while $\cup_{i=1}^m r([v_0]_1, \lambda_i)$ is finite for any finite number of labelled paths $\lambda_1, \ldots, \lambda_m$. Hence it is not possible to have $r([v_1]_1, x_{[1,N]}) \subset \cup_{i=1}^m r([v_0]_1, \lambda_i)$.

Remark 3.7. The ideal I generated by the projection $p_{\{v_0\}}$ in Example 3.6 is isomorphic to the graph C^*-algebra $C^*(E)$, where E is the following graph.



In fact, the elements

$$p_{v_n} := p_{\{v_n\}}, \;\; s_{e_n} := p_{\{v_{n-1}\}} s_b, \;\; s_{f_n} := p_{\{v_n\}} s_c, \;\; s_g := p_{\{v_0\}} s_a,$$

$n \in \mathbb{Z}$, generates the ideal I and forms a Cuntz-Krieger E-family. Since $C^*(E)$ is simple, we have $I \cong C^*(E)$.

Let $\{s_a, p_A\}$ be a universal representation of a labelled space $(E, \mathcal{L}, \mathcal{B})$ that generates $C^*(E, \mathcal{L}, \mathcal{B})$. Then

$$p_A s_\alpha \neq 0 \;\; \text{for } A \in \mathcal{B} \, (A \neq \emptyset), \;\; \alpha \in \mathcal{L}(AE^{\geq 1}). \tag{3}$$

In fact, if $p_A s_\alpha = s_\alpha p_{r(A,\alpha)} = 0$, then $s_\alpha^* s_\alpha p_{r(A,\alpha)} = p_{r(A,\alpha)} = 0$, but $p_{r(A,\alpha)}$ is nonzero since $r(A,\alpha) \in \mathcal{B}$ is non-empty.

Theorem 3.8. *Let $(E, \mathcal{L}, \overline{\mathcal{E}})$ be a weakly left-resolving labelled space. If $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is simple, then $(E, \mathcal{L}, \overline{\mathcal{E}})$ is strongly cofinal.*

Proof. Suppose that $(E, \mathcal{L}, \overline{\mathcal{E}})$ is not strongly cofinal. Then there are $[v]_l$, $x \in \mathcal{L}(E^\infty)$, and $w \in s(x)$ such that

$$r([w]_1, x_{[1,N]}) \nsubseteq \cup_{i=1}^m r([v]_l, \lambda_i) \tag{4}$$

for all $N \geq 1$ and any finite number of labelled paths $\lambda_1, \ldots, \lambda_m$. Consider the ideal I generated by the projection $p_{[v]_l}$. Suppose $p_{[w]_1} \in I$. Then there is an element $\sum_{j=1}^m c_j (s_{\alpha_j} p_{A_j} s_{\beta_j}^*) p_{[v]_l} (s_{\gamma_j} p_{B_j} s_{\delta_j}^*) \in I$, $c_j \in \mathbb{C}$, such that

$$\Big\| \sum_{j=1}^m c_j (s_{\alpha_j} p_{A_j} s_{\beta_j}^*) p_{[v]_l} (s_{\gamma_j} p_{B_j} s_{\delta_j}^*) - p_{[w]_1} \Big\| < 1. \tag{5}$$

By Remark 2.3, we may assume that the paths δ_j's in (5) have the same length. Then

$$\begin{aligned}
1 &> \Big\| \sum_j c_j (s_{\alpha_j} p_{A_j} s_{\beta_j}^*) p_{[v]_l} (s_{\gamma_j} p_{B_j} s_{\delta_j}^*) - p_{[w]_1} \Big\| \\
&\geq \Big\| \sum_j c_j (s_{\alpha_j} p_{A_j} s_{\beta_j}^*) p_{[v]_l} (s_{\gamma_j} p_{B_j} s_{\delta_j}^*) p_{[w]_1} - p_{[w]_1} \Big\| \\
&= \Big\| \sum_j c_j (s_{\alpha_j} p_{A_j} s_{\beta_j}^*) p_{[v]_l} (s_{\gamma_j} p_{r([v]_l, \gamma_j) \cap B_j \cap r([w]_1, \delta_j)} s_{\delta_j}^*) - p_{[w]_1} \Big\|.
\end{aligned}$$

We first show that for each $j = 1, \ldots, m$

$$r([w]_1, \delta_j) \subset \cup_{i=1}^m r([v]_l, \gamma_i). \tag{6}$$

Suppose $r([w]_1, \delta_j) \nsubseteq \cup_{i=1}^m r([v]_l, \gamma_i)$ for some j. Then $r([w]_1, \delta_j) \setminus \cup_{i=1}^m r([v]_l, \gamma_i) \in \overline{\mathcal{E}}$ is nonempty, hence

$$p_j := p_{r([w]_1, \delta_j) \setminus \cup_{i=1}^m r([v]_l, \gamma_i)} \neq 0.$$

Then with $J := \{ i \mid \delta_i = \delta_j \}$,

$$\begin{aligned}
1 &> \Big\| \Big(\sum_i c_i (s_{\alpha_i} p_{A_i} s_{\beta_i}^*) p_{[v]_l} (s_{\gamma_i} p_{r([v]_l, \gamma_i) \cap B_i \cap r([w]_1, \delta_i)} s_{\delta_i}^*) - p_{[w]_1} \Big) s_{\delta_j} \Big\| \\
&= \Big\| \sum_{i \in J} c_i (s_{\alpha_i} p_{A_i} s_{\beta_i}^*) p_{[v]_l} s_{\gamma_i} p_{r([v]_l, \gamma_i) \cap B_i \cap r([w]_1, \delta_i)} - p_{[w]_1} s_{\delta_j} \Big\| \\
&\quad \text{(here we use } |\delta_i| = |\delta_j|) \\
&= \Big\| \sum_{i \in J} c_i (s_{\alpha_i} p_{A_i} s_{\beta_i}^*) p_{[v]_l} s_{\gamma_i} p_{r([v]_l, \gamma_i) \cap B_i \cap r([w]_1, \delta_i)} - s_{\delta_j} p_{r([w]_1, \delta_j)} \Big\| \\
&\geq \Big\| \sum_{i \in J} c_i (s_{\alpha_i} p_{A_i} s_{\beta_i}^*) p_{[v]_l} s_{\gamma_i} p_{r([v]_l, \gamma_i) \cap B_i \cap r([w]_1, \delta_i)} p_j - s_{\delta_j} p_{r([w]_1, \delta_j)} p_j \Big\| \\
&= \Big\| s_{\delta_j} p_j \Big\| = 1,
\end{aligned}$$

which is a contradiction and (6) follows. Also $\delta_j \neq x_{[1,|\delta_j|]}$ for each j. In fact, if $\delta_j = x_{[1,|\delta_j|]}$, then by (6),

$$r([w]_1, x_{[1,|\delta_j|]}) = r([w]_1, \delta_j) \subseteq \cup_{j=1}^{m} r([v]_l, \gamma_j),$$

which contradicts to (4). Thus $s_{\delta_i}^*(s_{x_1}\cdots s_{x_N}) = 0$ for $i = 1, \ldots, m$, and $N = \max_{1\le j\le m}\{|\delta_j|\}$. Then $y := p_{[w]_1}s_{x_1}\cdots s_{x_N}$ is a nonzero partial isometry by (3) and $s_{\delta_i}^* y = s_{\delta_i}^* p_{[w]_1}s_{x_1}\cdots s_{x_N} = p_{r([w]_1,\delta_i)}s_{\delta_i}^*(s_{x_1}\cdots s_{x_N}) = 0$. From (5), we have

$$\begin{aligned} 1 &> \Big\| \, \Big(\sum_{i=1}^{m} \lambda_i (s_{\alpha_i}p_{A_i}s_{\beta_i}^*)p_{[v]_l}(s_{\gamma_i}p_{B_i}s_{\delta_i}^*)\Big)(yy^*) - p_{[w]_1}(yy^*) \, \Big\| \\ &= \|yy^*\| = 1, \end{aligned}$$

a contradiction, and so $p_{[w]_1} \notin I$. □

Recall [4] that $\alpha \in \mathcal{L}(E^{\ge 1})$ with $s(\alpha)\cap[v]_l \neq \emptyset$ is said to be *agreeable* for $[v]_l$ if $\alpha = \beta\alpha' = \alpha'\gamma$ for some α', β, $\gamma \in \mathcal{L}(E^{\ge 1})$ with $|\beta| = |\gamma| \le l$. Otherwise α is said to be *disagreeable*. We call $[v]_l$ *disagreeable* if there is an $N > 0$ such that for all $n > N$ there is an $\alpha \in \mathcal{L}(E^{\ge n})$ that is disagreeable for $[v]_l$. The labelled space $(E, \mathcal{L}, \mathcal{E}^{0,-})$ is *disagreeable* if for every $v \in E^0$ there is an $L_v > 0$ such that $[v]_l$ is disagreeable for all $l > L_v$. We say that a labelled space $(E, \mathcal{L}, \mathcal{B})$ is *disagreeable* if $(E, \mathcal{L}, \mathcal{E}^{0,-})$ is disagreeable.

Proposition 3.9. *Let $(E, \mathcal{L}, \mathcal{B})$ be a labelled space. Then we have the following.*

(i) *$[v]_l$ is not disagreeable if and only if there is an $N > 0$ such that every $\alpha \in \mathcal{L}(E^{\ge N})$ with $s(\alpha)\cap[v]_l \neq \emptyset$ is agreeable for $[v]_l$.*
(ii) *If $[v]_k$ is not disagreeable, $[v]_l$ is not disagreeable for all $l > k$.*
(iii) *$(E, \mathcal{L}, \mathcal{B})$ is disagreeable if and only if $[v]_l$ is disagreeable for all $l \ge 1$ and $v \in E^0$.*

Proof. (i) Note that $[v]_l$ is disagreeable if and only if there is a sequence $(\alpha_n)_{n=1}^{\infty} \subset \mathcal{L}(E^{\ge 1})$, $|\alpha_1| < |\alpha_2| < \cdots$, of labelled paths that are disagreeable for $[v]_l$. So $[v]_l$ is not disagreeable if and only if there is an $N > 0$ such that every labelled path α with $|\alpha| \ge N$ and $s(\alpha)\cap[v]_l \neq \emptyset$ is agreeable.

(ii) If $[v]_k$ is not disagreeable, then there is $N > 0$ satisfying the condition in (i). If $\alpha \in \mathcal{L}(E^{\ge N})$ and $s(\alpha)\cap[v]_l \neq \emptyset$, then $s(\alpha)\cap[v]_k \neq \emptyset$ (since $[v]_l \subset [v]_k$). Hence $\alpha = \beta\alpha' = \alpha'\gamma$ for some α', β, $\gamma \in \mathcal{L}(E^{\ge 1})$ with $|\beta| = |\gamma| \le k (< l)$, which means that $[v]_l$ is not disagreeable by (i).

(iii) By definition, $(E, \mathcal{L}, \mathcal{E}^{0,-})$ is disagreeable if $[v]_l$ is disagreeable for all $v \in E^0$ and $l \ge 1$. For the converse, let $(E, \mathcal{L}, \mathcal{E}^{0,-})$ be disagreeable. Suppose that $[v]_k$ is not disagreeable for some $v \in E^0$ and $k \ge 1$, then $[v]_l$ is not disagreeable for all $l > k$ by (ii), a contradiction. □

If α and α' are labelled paths such that either $\alpha = \alpha'$ or $\alpha = \alpha'\alpha''$ for some $\alpha'' \in \mathcal{L}(E^{\ge 1})$, we call α' an *initial segment* of α. If $\beta \in \mathcal{L}(E^{\ge 1})$, we

write β^∞ for the infinite labelled path $\beta\beta\cdots \in \mathcal{L}(E^\infty)$. We call $\beta \in \mathcal{L}(E^{\geq 1})$ *simple* if there is no labelled path $\delta \in \mathcal{L}(E^{\geq 1})$ such that $|\delta| < |\beta|$ and $\beta = \delta^n$ for some $n \geq 1$.

Remark 3.10. If γ and δ are simple labelled paths in $\mathcal{L}(vE^{\geq 1})$ such that $\gamma^{|\delta|} = \delta^{|\gamma|}$, then one can show that $\gamma = \delta$.

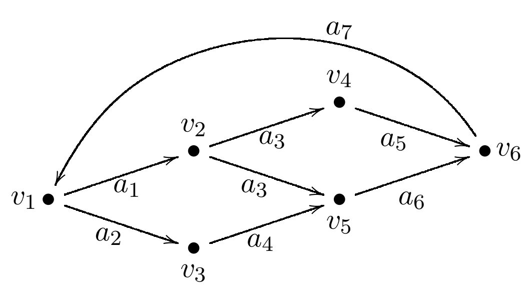
Lemma 3.11. *Let $(E, \mathcal{L}, \mathcal{B})$ be a labelled space and $v \in E^0$. If $[v]_l$ is not disagreeable, there exists an $N > 1$ such that every $\alpha \in \mathcal{L}([v]_l E^{\geq N})$ is of the form $\alpha = \beta^j \beta'$ for some $\beta \in \mathcal{L}(E^{\leq l})$ and an initial segment β' of β. Also every $x \in \mathcal{L}(E^\infty)$ with $s(x) \cap [v]_l \neq \emptyset$ is of the form $x = \beta^\infty$ for a simple labelled path $\beta \in \mathcal{L}(E^{\leq l})$.*

Proof. By Proposition 3.9(i) there is an $N > 1$ such that every $\alpha \in \mathcal{L}(E^{\geq N})$ with $s(\alpha) \cap [v]_l \neq \emptyset$, is of the form $\alpha = \beta\alpha' = \alpha'\gamma$ for some $\alpha', \beta, \gamma \in \mathcal{L}(E^{\geq 1})$ with $|\beta| = |\gamma| \leq l$. If $|\alpha'| \leq |\beta|$, then $\beta\alpha' = \alpha'\gamma$ shows that α' is an initial segment of β. If $|\alpha'| > |\beta|$, then $\alpha = \beta\alpha' = \alpha'\gamma$ implies that $\alpha' = \beta\alpha''$, hence $\alpha = \beta\alpha' = \beta^2\alpha'' = \beta\alpha''\gamma$. If $|\alpha''| > |\beta|$, we can repeat the argument until we get $\alpha = \beta^k\tilde{\alpha}$ for some $\tilde{\alpha}$ with $|\tilde{\alpha}| < \beta$. But we can always form a (longer) labelled path $\alpha\sigma$ extending α (with $|\sigma| \geq |\beta|$) and so the above argument shows that $\alpha\sigma = \beta^k\tilde{\alpha}\sigma$ must be of the form $\beta^l\sigma'$ for some σ' with $|\sigma'| < |\beta|$ $(l > k)$. Hence the path $\tilde{\alpha}$ is an initial segment of β and we prove the assertion.

Let $x \in \mathcal{L}(E^\infty)$ and $s(x) \cap [v]_l \neq \emptyset$. Then $x_{[1,N+k]} = \beta_{(k)}^{m_k}\beta'_{(k)}$ for a simple labelled path $\beta_{(k)} \in \mathcal{L}(E^{\leq l})$ and its initial segment $\beta'_{(k)}$. Note that if $|\beta_{(k)}| \leq |\beta_{(k')}|$, $\beta_{(k)}$ is an initial path of $\beta_{(k')}$. Since $|\beta_{(k)}| \leq l$ for all k, there is a subsequence $(\beta_{(k_j)})_j$ of $(\beta_{(k)})_k$ such that $|\beta_{(k_j)}| = |\beta_{(k_1)}|$ for all $j \geq 1$. Then with $\beta := \beta_{(k_1)}$ we finally have $x = \beta^\infty$. $\square$

If a labelled space $(E, \mathcal{L}, \overline{\mathcal{E}})$ is weakly left-resolving and $v \in E^0$, then $\{v\} \in \overline{\mathcal{E}}$ if and only if $[v]_l = \{v\}$ for some $l \geq 1$. In Example 3.6, $\{v\} \in \overline{\mathcal{E}}$ for every $v \in E^0$. Also the following example shows that the condition $\{v\} \in \overline{\mathcal{E}}$ for every $v \in E^0$ does not imply that $\overline{\mathcal{E}} = \mathcal{E}^{0,-}$.

Example 3.12. In the following labelled graph $(E, \mathcal{L})$, we have $\{v_i\} \in \overline{\mathcal{E}}$ for every $i = 1, \ldots, 6$, but $\{v_4\} \notin \mathcal{E}^{0,-}$.



For a labelled path $\beta \in \mathcal{L}(E^{\geq 1})$, put $T(\beta) := \beta_2\beta_3\cdots\beta_{|\beta|}\beta_1$ whenever $\beta_2\beta_3\cdots\beta_{|\beta|}\beta_1 \in \mathcal{L}(E^{\geq 1})$. We write $T^k(\beta)$ for $T(T^{k-1}(\beta))$, $k \geq 2$, and put $T^0(\beta) := \beta$.

Lemma 3.13. *Let $(E, \mathcal{L}, \overline{\mathcal{E}})$ be a weakly left-resolving and strongly cofinal labelled space such that $\{w\} \in \overline{\mathcal{E}}$ for every $w \in E^0$. If $[v]_l$ is not disagreeable, we have the following.*

(a) *$\mathcal{L}(vE^\infty) = \{\beta^\infty\}$ for a simple labelled path $\beta \in \mathcal{L}(E^{\leq l})$.*

(b) *$|\, r(\{v\}, \alpha)\,| = 1$ for every $\alpha \in \mathcal{L}(vE^{\geq 1})$.*

Proof. (a) By Lemma 3.11, every $x \in \mathcal{L}(vE^\infty)$ is of the form $x = \beta^\infty$ for a simple labelled path $\beta \in \mathcal{L}(E^{\leq l})$.

Claim(I) Let $x, y \in \mathcal{L}(vE^\infty)$ and $x = \beta^\infty$, $y = \gamma^\infty$ for simple labelled paths $\beta, \gamma \in \mathcal{L}(vE^{\leq l})$. Then $\gamma = T^k(\beta)$ for some $k \geq 0$ (thus $|\beta| = |\gamma|$).

To prove Claim(I), let $x = x_1x_2\cdots$ and $y = y_1y_2\cdots$, x_i, $y_i \in \mathcal{A}$. Suppoe there is an $N \geq 1$ such that $r(\{v\}, x_{[1,j]}) = r(\{v\}, y_{[1,j]})$ for all $j \geq N$. Choose $m \geq 1$ with $m|\beta||\gamma| > N$ and consider the infinite labelled path $z := x_{[\,1,m|\beta||\gamma|\,]}y_{[\,m|\beta||\gamma|+1,\infty)} \in \mathcal{L}(vE^\infty)$. Then z must be of the form $z = \sigma^\infty$ for a simple labelled path $\sigma \in \mathcal{L}(vE^{\leq l})$ and so

$$z = \beta^{m|\gamma|}\gamma^\infty = \sigma^\infty. \tag{7}$$

Since γ and σ are simple, it follows that $\sigma = T^k(\gamma)$ for some $k \geq 0$. Then $|\sigma| = |\gamma|$ and we have $\beta^{m|\gamma|} = \beta^{m|\sigma|} = \sigma^{m|\beta|}$ from (7). By Remark 3.10, it follows that $\beta = \sigma = T^k(\gamma)$. Thus $|\beta| = |\gamma| = |\sigma|$. Then (7) shows $\beta = \gamma$. Thus if $\beta \neq \gamma$, we may assume that there exists a vertex $w \in r(\{v\}, y_{[1,j]}) \setminus r(\{v\}, x_{[1,j]})$ for some j large enough with $j > |\beta||\gamma|$. Since $(E, \mathcal{L}, \overline{\mathcal{E}})$ is strongly cofinal, there is an N_1 and a finite number of labelled paths $\gamma_1, \ldots, \gamma_m$ such that

$$r(\{v\}, x_{[1,N_1]}) \subset r([v]_1, x_{[1,N_1]}) \subset \cup_{i=1}^m r(\{w\}, \gamma_i).$$

If $r(\{v\}, x_{[1,N_1]}) \cap r(\{w\}, \gamma_i) \neq \emptyset$, then the labelled path $z := y_{[1,j]}\gamma_i x_{[N_1+1,\infty)}$ must be of the form $z = \sigma^\infty$ for a simple labelled path $\sigma \in \mathcal{L}(vE^{\leq l})$. Thus

$$z = \sigma^\infty = y_{[1,j]}\gamma_i x_{N_1+1}x_{N_1+2}\cdots\beta^\infty,$$

and so $\sigma = T^k(\beta)$ for some $k \geq 0$ because σ and β are simple. Since the initial segment $y_{[1,j]}$ of z has length $j > |\beta||\gamma|$, z must be of the form

$$z = \sigma^{|\gamma|}\sigma^\infty = \gamma^{|\sigma|}\cdots\sigma^\infty,$$

hence $\sigma^{|\gamma|} = \gamma^{|\sigma|}$. Then by Remark 3.10, $\sigma = \gamma$. Thus $\gamma = T^k(\beta)$ for some $k \geq 0$, and Claim(I) is proved.

Claim(II) If $x = \beta^\infty$, $y = \gamma^\infty \in \mathcal{L}(vE^\infty)$ for simple labelled paths $\beta, \gamma \in \mathcal{L}(vE^{\leq l})$, then $\beta = \gamma$.

Suppose $\beta \neq \gamma$. Then by Claim(I), $\gamma = T^k(\beta)$ for some $k \geq 1$. Let $m = |\beta| = |\gamma|$. By the first argument in the proof of Claim(I), we may assume

that there is a vertex $u \in r(\{v\}, \gamma^n\gamma_{[1,j]}) \setminus r(\{v\}, \beta^n\beta_{[1,j]})$ for some $n \geq 0$ and $0 \leq j \leq n-1$, here $\gamma^n\gamma_{[1,0]} := \gamma^n$ and u can be chosen as $u \neq v$. By strong cofinality, there exist $\delta = \delta_1 \cdots \delta_{|\delta|} \in \mathcal{L}(E^{\geq 1})$ and $N \geq 1$ such that

$$r(\{v\}, x_{[1,N]}) \cap r(\{u\}, \delta) \neq \emptyset$$

and

$$r(\{v\}, x_{[1,N-1]}) \cap r(\{u\}, \delta_{[1,j]}) = \emptyset \ \text{ for all } 0 \leq j < |\delta|. \tag{8}$$

(Here $r(\{u\}, \delta_{[1,0]}) := \{u\}$.) Since $x = \beta^\infty$, we can write

$$x = x_1x_2 \cdots x_N \cdots = \beta\beta \cdots x_N\beta_{[i,m]}\beta^\infty$$

for some i. Now consider the labelled path

$$\tilde{y} := \gamma^n\gamma_{[1,j]}\delta\beta_{[i,m]}\beta^\infty \in \mathcal{L}(vE^\infty).$$

By Claim(I), $\tilde{y} = T^k(\beta)^\infty$ for some $k \geq 1$, hence $x_N\beta_{[i,m]}\beta^\infty = \delta_{|\delta|}\beta_{[i,m]}\beta^\infty$ and we have $x_N = \delta_{|\delta|}$. Then

$$\begin{aligned} \emptyset &\neq r(\{v\}, x_{[1,N]}) \cap r(\{u\}, \delta) \\ &= r(r(\{v\}, x_{[1,N-1]}), x_N) \cap r(r(\{u\}, \delta_{[1,|\delta|-1]}), \delta_{|\delta|}) \\ &= r(r(\{v\}, x_{[1,N-1]}), x_N) \cap r(r(\{u\}, \delta_{[1,|\delta|-1]}), x_N). \end{aligned}$$

But $r(\{v\}, x_{[1,N-1]}) \cap r(\{u\}, \delta_{[1,|\delta|-1]}) = \emptyset$ by (8), a contradiction to that the labelled space is weakly left-resolving, and Claim(II) is proved.

(b) Suppose $|\,r(\{v\}, \alpha)\,| > 1$ for an $\alpha \in \mathcal{L}(vE^{\geq 1})$. Then there are two paths $\mu, \nu \in E^{\geq 1}$ with $s(\mu) = s(\nu) = v$ and $\alpha = \mathcal{L}(\mu) = \mathcal{L}(\nu)$ such that $v_1 := r(\mu)$ and $v_2 := r(\nu)$ are distinct. Let $y \in \mathcal{L}(v_1E^\infty)$. Since $(E, \mathcal{L}, \overline{\mathcal{E}})$ is strongly cofinal, there exist $N \geq 1$ and $\lambda_1, \ldots, \lambda_n \in \mathcal{L}(E^{\geq 1})$ such that

$$r([v_1]_1, y_{[1,N]}) \subset \cup_{i=1}^n r(\{v_2\}, \lambda_i).$$

We can choose $\lambda \in \{\lambda_1, \ldots, \lambda_n\}$ with $r([v_1]_1, y_{[1,N]}) \cap r(\{v_2\}, \lambda) \neq \emptyset$ and may assume that

$$r([v_1]_1, y_{[1,j]}) \cap r(\{v_2\}, \lambda_{[1,i]}) = \emptyset \tag{9}$$

for all $1 \leq j \leq N$ and $0 \leq i < |\lambda|$. Since both αy and $\alpha\lambda y_{[N+1,\infty]}$ belong to $\mathcal{L}(vE^\infty)$, by (a)

$$\alpha y = \alpha\lambda y_{[N+1,\infty]} = \sigma^\infty$$

for a simple labelled path $\sigma \in \mathcal{L}(vE^{\leq l})$. Thus we have

$$y = \lambda y_{[N+1,\infty]} = T^k(\sigma)^\infty,$$

for some k, and we obtain $y_N = \lambda_{|\lambda|}$. Note that

$$\begin{aligned} &r(r(\{v_1\}, y_{[1,N-1]}), y_N) \cap r(r(\{v_2\}, \lambda_{|\lambda|-1}), \lambda_{|\lambda|}) \\ &= r(\{v_1\}, y_{[1,N]}) \cap r(\{v_2\}, \lambda) \\ &\neq \emptyset \end{aligned}$$

while $r(\{v_1\}, y_{[1,N-1]}) \cap r(\{v_2\}, \lambda_{|\lambda|-1}) = \emptyset$ by (9), a contradiction to the assumption that $(E, \mathcal{L}, \overline{\mathcal{E}})$ is weakly left-resolving. $\square$

Theorem 3.14. *Let $(E, \mathcal{L}, \overline{\mathcal{E}})$ be a weakly left-resolving labelled space such that $\{v\} \in \overline{\mathcal{E}}$ for each $v \in E^0$. If $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is simple, then $(E, \mathcal{L}, \overline{\mathcal{E}})$ is disagreeable.*

Proof. By Theorem 3.8, $(E, \mathcal{L}, \overline{\mathcal{E}})$ is strongly cofinal. Suppose that $(E, \mathcal{L}, \overline{\mathcal{E}})$ is not disagreeable. Then there exists $v \in E^0$ and $l \geq 1$ such that $[v]_l$ is not disagreeable by Proposition 3.9(iii). Since $\{v\} \in \overline{\mathcal{E}}$, by Proposition 3.9(ii) and Proposition 3.4 we may assume that $[v]_l = \{v\}$. Then, by Lemma 3.13, $\mathcal{L}(vE^\infty) = \{\beta^\infty\}$ for a simple labelled path $\beta \in \mathcal{L}(E^{\leq l})$ and

$$|\, r(\{v\}, \alpha) \,| = 1 \quad \text{for every} \quad \alpha \in \mathcal{L}(vE^{\geq 1}). \tag{10}$$

Now we consider two possible cases (1) and (2).

Case(1) There is a loop $\mu \in E^{\geq 1}$ at a vertex $w \in \{v\} \cup r(\mathcal{L}(vE^{\geq 1}))$. We may assume that $\mu = \mu_1 \cdots \mu_{|\mu|}$ is a simple loop, that is, $r(\mu_i) \neq r(\mu_j)$ for $i \neq j$. Note from Assumption 2.2 and (10) that μ has no exits and there are vertices $u_j \in E^0$, $j = 1, \ldots, |\mu|$, such that

$$r(\{w\}, \mathcal{L}(\mu)_{[1,j]}) = \{u_j\}.$$

Let $A := \{u_1, \ldots, u_{|\mu|}\}$. Then A and $\{u_j\}$ belong to $\overline{\mathcal{E}}$ so that the projections p_A and $p_j := p_{\{u_j\}}$, $j = 1, \ldots, |\mu|$, are nonzero and p_A is the unit of the C^*-subalgebra $p_A C^*(E, \mathcal{L}, \overline{\mathcal{E}}) p_A$ which is simple as a hereditary C^*-subalgebra of a simple C^*-algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$. For $\gamma, \delta \in \mathcal{L}(E^{\geq 1})$, note from (10) that

$$r(A, \gamma) \cap r(A, \delta) \neq \emptyset \Longleftrightarrow r(A, \gamma) = r(A, \delta) = \{u_j\},\ j = 1, \ldots, |\mu|. \tag{11}$$

Also for $s_\gamma p_B s_\delta^* \in C^*(E, \mathcal{L}, \overline{\mathcal{E}})$, $\gamma, \delta \in \mathcal{L}(E^{\geq 1})$ and $B \in \overline{\mathcal{E}}$, if $p_A(s_\gamma p_B s_\delta^*) p_A = s_\gamma p_{r(A,\gamma) \cap B \cap r(A,\delta)} s_\delta^* \neq 0$, then $s_\gamma p_{r(A,\gamma)} s_\delta^* = s_\gamma p_j s_\delta^* \neq 0$ for some j. Thus we have

$$\begin{aligned} p_A C^*(E, \mathcal{L}, \overline{\mathcal{E}}) p_A &= \overline{span}\{p_A(s_\gamma p_B s_\delta^*) p_A \mid \gamma, \delta \in \mathcal{L}(E^{\geq 1}),\ B \in \overline{\mathcal{E}}\,\} \\ &= \overline{span}\{s_\gamma p_j s_\delta^*, \mid \gamma, \delta \in \mathcal{L}(AE^{\geq 1} u_j),\ j = 1, 2, \ldots, |\mu|\,\}. \end{aligned}$$

But, since $p_j = \sum_{a \in \mathcal{L}(u_j E^1)} s_a p_{r(u_j, a)} s_a^*$ and $\mathcal{L}(u_j E^1) = \{\mathcal{L}(\mu_{j+1})\}$,

$$\begin{aligned} s_\gamma p_j &= s_\gamma s_{\mathcal{L}(\mu_{j+1})} p_{r(u_j, \mathcal{L}(\mu_{j+1}))} s^*_{\mathcal{L}(\mu_{j+1})} \\ &= s_{\gamma \mathcal{L}(\mu_{j+1})} p_{j+1} s^*_{\mathcal{L}(\mu_{j+1})} \end{aligned}$$

for $\gamma \in \mathcal{L}(AE^{\geq 1} u_j)$ and $j = 1, \ldots, |\mu|$ (here $j+1$ means 1 if $j = |\mu|$), that is, $s_\gamma p_j \in p_A C^*(E, \mathcal{L}, \overline{\mathcal{E}}) p_A$. Also every $\gamma \in \mathcal{L}(AE^{\geq 1} u_j)$ satisfies $\gamma = p_{j-1} \gamma p_j$, for some $j = 1, \cdots, |\mu|$, where $p_0 := p_{|\mu|} = p_w$. Thus $p_A C^*(E, \mathcal{L}, \overline{\mathcal{E}}) p_A$ is the C^*-algebra generated by the nonzero partial isometries $s_j := p_{j-1} s_{\mathcal{L}(\mu_j)} p_j$, $j = 1, \cdots, |\mu|$ such that

$$s_j^* s_j = s_{j+1} s_{j+1}^*,\ \ s_i^* s_j = 0\ (i \neq j),\ \text{and}\ \ \sum_{j=1}^{|\mu|} s_j^* s_j = p_A.$$

Hence it is a quotient algebra of $C(\mathbb{T})\otimes M_{|\mu|}$ which is the graph C^*-algebra of the graph with the vertices $r(\mu_i)$ and the edges μ_i, $i=1,\dots,|\mu|$. Considering the restriction of the gauge action γ_z, $z\in\mathbb{Z}$, on $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ to $p_AC^*(E,\mathcal{L},\overline{\mathcal{E}})p_A$, we see by the gauge invariant uniqueness theorem (see [3, Theorem 5.3]) that $p_AC^*(E,\mathcal{L},\overline{\mathcal{E}})p_A\cong C(\mathbb{T})\otimes M_{|\mu|}$, a contradiction.

Case(2) Suppose that there is no loop at a vertex in $\{v\}\cup r(\mathcal{L}(vE^{\geq 1}))$. Recall that $\mathcal{L}(vE^\infty)=\{\beta^\infty\}$ for a simple labelled path $\beta\in\mathcal{L}(E^{\leq l})$. Then $r(\{v\},\beta^m\beta_{[1,j]})\neq r(\{v\},\beta^n\beta_{[1,k]})$ for $m\neq n$ or $j\neq k$. Since $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ is simple, if I denotes the ideal generated by the projection $p_{\{v\}}$, then $I=C^*(E,\mathcal{L},\overline{\mathcal{E}})$. Hence there exists $X\in I$ such that $\|s_\beta^*s_\beta-X\|<\frac{1}{2}$. Write $X=\sum_{i=1}^m\lambda_i(s_{\alpha_i}p_{A_i}s_{\sigma_i}^*)p_{\{v\}}(s_{\gamma_i}p_{B_i}s_{\delta_i}^*)$, $\lambda_i\in\mathbb{C}$. Since

$$(s_{\alpha_i}p_{A_i}s_{\sigma_i}^*)p_{\{v\}}(s_{\gamma_i}p_{B_i}s_{\delta_i}^*)=s_{\alpha_i}p_{A_i}p_{r(\{v\},\sigma_i)}s_{\sigma_i}^*s_{\gamma_i}p_{r(\{v\},\gamma_i)}p_{B_i}s_{\delta_i}^*,$$

by Lemma 3.11 we may assume that σ_i's and γ_i's are of the form $\beta^n\beta_{[1,k]}$. Choose $N_1>0$ large enough so that for every $x\in\mathcal{L}(E^{\geq N_1})$, the range vertex set $r(\{v\},x)$ dose not meet $r(\{v\},\sigma_i)$ or $r(\{v\},\gamma_i)$ for all $i=1,\dots,m$. Then with $\{u\}=r(\{v\},\beta^{N_1})$ (recall that $r(\{v\},\beta^{N_1})$ is a singleton set),

$$p_{\{u\}}s_{\alpha_i}p_{r(\{v\},\sigma_i)}=s_{\alpha_i}p_{r(\{u\},\alpha_i)}p_{r(\{v\},\sigma_i)}=s_{\alpha_i}p_{r(\{v\},\,\beta^{N_1}\alpha_i)}p_{r(\{v\},\sigma_i)}=0$$

for all $i=1,\dots,m$ (since $|\beta^{N_1}\alpha_i|>|\sigma_i|$), and so we obtain

$$\frac{1}{2}>\ \|p_{\{u\}}(s_\beta^*s_\beta-X)\|=\|p_{\{u\}}p_{r(\beta)}p_{\{u\}}\|=\|p_{\{u\}}\|=1,$$

a contradiction. □

Remark 3.15. Let $\pi_{S,P}$ be a non-zero representation of $C^*(E,\mathcal{L},\mathcal{E})$, where $\mathcal{E}=\overline{\mathcal{E}}$ or $\mathcal{E}^{0,-}$. Consider a generalized vertex $[w]_d$ for which $P_{[w]_d}$ is a nonzero projection in $C^*(E,\mathcal{L},\mathcal{E})$. Since $[w]_d$ is the disjoint union of a finite number of equivalence classes $[w_i]_k$ whenever $k\geq d$, for each k there is an i such that $P_{[w_i]_k}\neq 0$ as noted in the proof of [4, Theorem 6.4]. But it does not mean that we may assume $P_{[w]_d}\neq 0$ for $d\geq R(w)$ as claimed there. For example, consider the labelled graph C^*-algebra $C^*(E,\mathcal{L},\mathcal{E})$ in Example 3.6 and the ideal I generated by the projection $p_{\{v_0\}}$. Let $\pi: C^*(E,\mathcal{L},\mathcal{E})\to C^*(E,\mathcal{L},\mathcal{E})/I$ be the quotient map (see Remark 3.7). Then

$$\pi(p_{[v_n]_1})=\pi(p_{E^0\setminus\{v_0\}})\neq 0\ \text{ for } n\neq 0$$

but $\pi(p_{[v_n]_d})=\pi(p_{\{v_n\}})=0$ for $d\geq R(v_n)(\geq 2)$ and $n\neq 0$.

Nevertheless, if we assume that $(E,\mathcal{L},\mathcal{E})$ is strongly cofinal and disagreeable, a slight modification of the proof of [4, Theorem 6.4 and Theorem 5.5] gives the following theorem.

Theorem 3.16. *Let $(E,\mathcal{L},\mathcal{E})$ be a labelled space that is strongly cofinal and disagreeable, where $\mathcal{E}=\overline{\mathcal{E}}$ or $\mathcal{E}^{0,-}$. Then $C^*(E,\mathcal{L},\mathcal{E})$ is simple.*

Corollary 3.17. *Let $(E, \mathcal{L}, \mathcal{E})$ be a labelled space such that $\{v\} \in \mathcal{E}$ for each $v \in E^0$, where $\mathcal{E} = \overline{\mathcal{E}}$ or $\mathcal{E}^{0,-}$. Then $C^*(E, \mathcal{L}, \mathcal{E})$ is simple if and only if $(E, \mathcal{L}, \mathcal{E})$ is strongly cofinal and disagreeable.*

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